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INSTABILITY MODES OF CANTILEVERED BARS INDUCED BY FLUID FLOW THROUGH ATTACHED PIPES*

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Abstract—A cantilevered bar of uniform cross-section with flexible pipes conveying fluid attached to it is considered. It is shown that for certain cross-sections of the bar stability may be lost by either torsional divergence (torsional buckling) or torsional flutter, depending upon the location of the pipes with respect to the center of gravity of the cross-section. In addition, transverse flutter can also occur, but not transverse buckling. It is shown further that the Coriolis forces, which are present due to the motion of the fluid in vibrating pipes, may have either a stabilizing or a destabilizing effect, depending upon the parameters of the system.

1. INTRODUCTION

IN a recent paper [1], the present authors have shown that a cantilevered bar subjected at the free end to distributed, compressive, follower forces may lose stability by either torsional divergence (torsional buckling) or torsional flutter (torsional oscillations with increasing amplitude), depending upon the load distribution at the end section. In addition, it was shown that transverse flutter can also occur, but not transverse divergence. However, in [1], no attempt was made to indicate how such systems may be realized.

The purpose of the present study is to show that a cantilevered bar having two axes of symmetry can exhibit all the features outlined in [1] when two pairs of flexible pipes are placed symmetrically at the distance $h/2$ from the longitudinal axis of the bar (z-axis, Fig. 1), and an incompressible fluid at a constant velocity U is pumped through the pipes. In the sequel, we shall see that, in addition to transverse flutter, the system can exhibit both torsional buckling and torsional flutter, depending upon the value of h. Moreover, the Coriolis forces which are now present due to the motion of the fluid in vibrating pipes, can have either a stabilizing or a destabilizing effect. That is, the Coriolis forces, similar to viscous damping forces, can either increase or decrease the critical flutter load (both in torsional flutter and transverse flutter of the system), depending upon the parameters of the system.

The destabilizing effect of velocity-dependent forces in nonconservative problems was formally established by the authors in [2] for systems with a finite number of degrees of freedom. In particular it was shown that the critical load of the undamped system (no velocity-dependent forces exist) is an upper bound for the critical load of the same system when some sufficiently small velocity-dependent forces are also present.

In the past, many authors [3–6] have concluded that viscous damping may have a destabilizing effect in nonconservative continuous systems. However, this conclusion was

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obtained by reducing the continuous system to a system with a finite number of degrees of freedom which, as was shown in [2], exhibits a destabilizing effect. Therefore, one does not know whether the destabilizing effect observed has been produced through the reduction of the system with an infinite number of degrees of freedom or is an inherent property of the continuous system. This difficulty is circumvented by the authors in [7]. A cantilevered pipe conveying fluid is considered there and the internal and external viscous damping forces are also included. It is then shown that sufficiently small Coriolis forces and viscous damping forces may, in fact, have a destabilizing effect in this system which has an infinite number of degrees of freedom.

In the present study we shall show that the above effect is also present in torsional flutter of the system under study. Moreover, even for relatively large values of Coriolis forces, the destabilizing effect may still exist for some values of h.

2. DERIVATION OF EQUATION OF MOTION AND BOUNDARY CONDITIONS

We consider a thin-walled, cantilevered, elastic beam with two pairs of flexible pipes, which are attached to the bar at a distance $h/2$ from the z-axis (so that the whole system deforms as a unit) and pump fluid at a constant velocity U through the pipes, as sketched in Fig. 1. We designate the length of the system by L, the torsional rigidity by $C = GJ$,

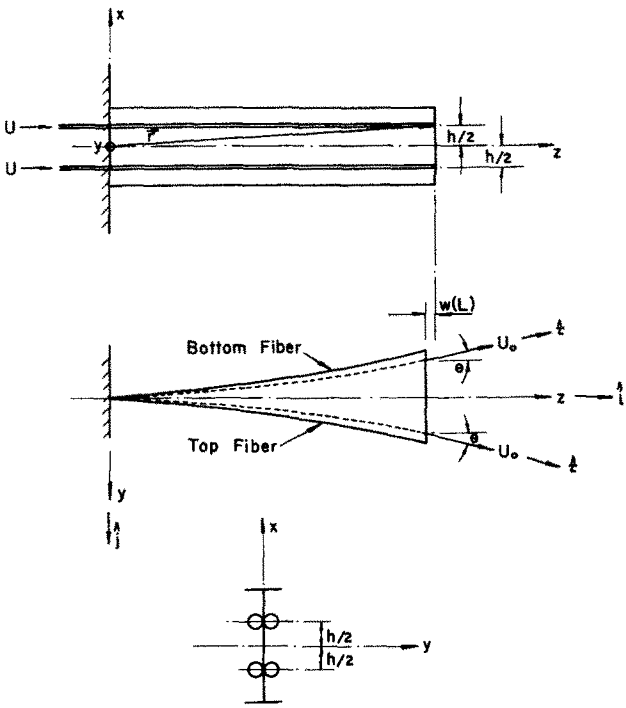

FIG. 1

and the warping rigidity by $C_1 = EC_W$ [8], and similar to the work of Benjamin [9] obtain the equation of torsional motion of the system, using Hamilton's principle. With $\varphi(z, t)$ denoting the angle of rotation at section z and at time t, the strain energy of the torsional deformation is [10]

$$V_1 = \tfrac{1}{2}\int_0^L [C_1(\varphi'')^2 + C(\varphi')^2]\,\mathrm{d}z, \tag{1}$$

where primes denote differentiation with respect to z. The kinetic energy is

$$T_1 = \tfrac{1}{2}\int_0^L mr^2(\dot{\varphi})^2\,\mathrm{d}z, \tag{2}$$

where a dot denotes differentiation with respect to time, m is the mass of the assembly per unit of length (exclusive of the mass of the fluid), and r is the polar radius of gyration of the cross-section of the system.

The total kinetic energy of the fluid may be obtained by adding to the kinetic energy of the fluid contained within the pipes, T_2, the change in the kinetic energy of the fluid entering and leaving the pipes during a very small interval of time Δt:

$$T' = T_2 + 2MU(\tfrac{1}{2}U_o^2 - \tfrac{1}{2}U_i^2)\,\Delta t, \tag{3}$$

where T' is the total kinetic energy of the fluid, M the mass density of the fluid per unit length of each pair of pipes, $\mathbf{U}_o$ the outlet velocity vector, and $\mathbf{U}_i$ the inlet velocity vector. But $\mathbf{U}_i = U\hat{\imath}$, where $\hat{\imath}$ is the unit vector in the z-direction, and $\mathbf{U}_o = \dot{\mathbf{r}} + nU\hat{\tau}$ where $\hat{\tau}$ is the unit vector tangent to the top (bottom) pipe at $z = L$, $\mathbf{r}$ the position vector of the top (bottom) pipe at $z = L$, and n is the ratio of the area of each pipe to that of the attached nozzle at the free end. Hence, $\delta T'$ becomes

$$\delta T' = \delta T_2 + 2MU(\dot{\mathbf{r}} + nU\hat{\tau})\,.\,\delta\mathbf{r}. \tag{4}$$

The components of the absolute velocity of the fluid are $\dot{y} + U(\partial y/\partial z)$ in the y-direction, and $U[1 - \tfrac{1}{2}(y')^2] - \dot{w}$ in the z-direction, where $w(z, t)$ denotes the average displacement at section z and at time t in the z-direction. T_2 then becomes (within an additive constant)

$$T_2 = 2M\int_0^L (\tfrac{1}{2}\dot{y}^2 + U\dot{y}y' - U\dot{w})\,\mathrm{d}z.$$

But $y = (h/2)\varphi$, which yields

$$T_2 = 2M\int_0^L \left[\frac{h^2}{8}\dot{\varphi}^2 + \frac{Uh^2}{4}\dot{\varphi}\varphi' - U\dot{w}\right]\mathrm{d}z. \tag{5}$$

With $\hat{\jmath}$ being the unit vector along the y-axis, we have (see Fig. 1)

$$\begin{aligned}\hat{\tau} &= \hat{\jmath}\sin\theta + \hat{\imath}\cos\theta = \hat{\jmath}(y')_{z=L} + \hat{\imath}\\ &= \left[\frac{h}{2}\varphi'(L)\right]\hat{\jmath} + \hat{\imath},\\ \mathbf{r} &= \hat{\jmath}(y)_{z=L} - \hat{\imath}(w)_{z=L} = \left[\frac{h}{2}\varphi(L)\right]\hat{\jmath} - [w(L)]\hat{\imath}.\end{aligned}$$

Then

$$(\dot{\mathbf{r}} + nU\hat{\tau})\,.\,\delta\mathbf{r} \approx -nU\,\delta w(L) + \frac{h^2}{4}[\dot{\varphi}(L) + nU\varphi'(L)]\,\delta\varphi(L), \tag{6}$$

where $\dot{w}(L)\,\delta w(L)$ is neglected (being a term of higher order). The Lagrangian now becomes

$$L = T_1 + T_2 - V_1 + 2MnU^2 w(L) \tag{7}$$

and Hamilton's principle takes on the form

$$\delta \int_{t_1}^{t_2} L\,\mathrm{d}t - \int_{t_1}^{t_2} MU\frac{h^2}{2}[\dot{\varphi}(L) + nU\varphi'(L)]\,\delta\varphi(L)\,\mathrm{d}t = 0, \tag{8}$$

where

$$w(L) = \tfrac{1}{2}\int_0^L r^2(\varphi')^2\,\mathrm{d}z.$$

Carrying out the variations and using integration by parts, we obtain

$$C_1\frac{\partial^4\varphi}{\partial z^4} + [2MnU^2r^2 - C]\frac{\partial^2\varphi}{\partial z^2} + MUh^2\frac{\partial^2\varphi}{\partial z\partial t} + \left(mr^2 + M\frac{h^2}{2}\right)\frac{\partial^2\varphi}{\partial t^2} = 0,$$

$$\varphi = \frac{\partial\varphi}{\partial z} = 0; \qquad z = 0,$$

$$\left.\begin{aligned} \frac{\partial^2\varphi}{\partial z^2} &= 0, \\ C_1\frac{\partial^3\varphi}{\partial z^3} + \left[MnU^2\left(2r^2 - \frac{h^2}{2}\right) - C\right]\frac{\partial\varphi}{\partial z} &= 0 \end{aligned}\right\}; \qquad z = L. \tag{9}$$

We now introduce the following dimensionless quantities:

$$\xi = \frac{z}{L},$$

$$\tau = t\sqrt{\left[\frac{C_1}{[mr^2 + (h^2/2)M]L^4}\right]},$$

$$\kappa = \frac{CL^2}{C_1},$$

$$\alpha = \frac{h}{r}, \qquad \beta = \frac{M}{m},$$

$$\beta' = \frac{\beta}{1 + (\alpha^2/2)\beta}, \quad \text{and} \quad F = \frac{MnU^2r^2L^2}{C_1}.$$

Equations (9) then become

$$\frac{\partial^4\varphi}{\partial\xi^4}+[2F-\kappa]\frac{\partial^2\varphi}{\partial\xi^2}+\sqrt{\left(\beta'\frac{F}{n}\right)}\alpha^2\frac{\partial^2\varphi}{\partial\xi\partial\tau}+\frac{\partial^2\varphi}{\partial\tau^2}=0,$$

$$\varphi=\frac{\partial\varphi}{\partial\xi}=0;\quad \text{at}\quad \xi=0,$$

$$\left.\begin{aligned}&\frac{\partial^2\varphi}{\partial\xi^2}=0,\\ &\frac{\partial^3\varphi}{\partial\xi^3}+\left[F\left(2-\frac{\alpha^2}{2}\right)-\kappa\right]\frac{\partial\varphi}{\partial y}=0\end{aligned}\right\};\qquad \xi=1 \tag{10}$$

which are analogous to those obtained by the present authors in [1] for cantilevered bars subjected at the free end to follower forces, except for the third term in the first equation, which is due to the Coriolis acceleration. As we shall see in the sequel, this term can have either a destabilizing or a stabilizing effect. That is, for sufficiently small Coriolis forces (n large and β' small) the system loses stability (by torsional or transverse flutter) under smaller F than obtained when $n=\infty$ (no Coriolis forces). On the other hand, for β'/n sufficiently large, the critical value of F can be increased by increasing β'/n.

We note here that, in torsional instability similar to transverse instability, the Coriolis forces have an effect similar to that of internal viscous damping [7]. That is, although damping (and also Coriolis forces) is a dissipating agency, when it is sufficiently small, it may act as a channel for the transfer of energy to the system from the source, which is always associated with the type of nonconservative forces considered here [11].

3. STABILITY ANALYSIS

3.1 *Frequency equation*

We take the solution of system (10) as $\varphi(\xi,\tau)=\psi(\xi)e^{i\omega\tau}$ and obtain the following eigenvalue problem:

$$\frac{d^4\psi}{d\xi^4}+[2F-\kappa]\frac{d^2\psi}{d\xi^2}+(i\omega)\sqrt{\left(\beta'\frac{F}{n}\right)}\alpha^2\frac{d\psi}{d\xi}-\omega^2\psi=0,$$

$$\psi=\frac{d\psi}{d\xi}=0;\quad \text{at}\quad \xi=0,$$

$$\left.\begin{aligned}&\frac{d^2\psi}{d\xi^2}=0,\\ &\frac{d^3\psi}{d\xi^3}+\left[F\left(2-\frac{\alpha^2}{2}\right)-\kappa\right]\frac{d\psi}{d\xi}=0\end{aligned}\right\};\quad \xi=1 \tag{11}$$

We then let $\psi(\xi)=Ae^{i\lambda\xi}$ and obtain

$$\lambda^4-(2F-\kappa)\lambda^2-\omega\sqrt{\left(\beta'\frac{F}{n}\right)}\alpha^2\lambda-\omega^2=0. \tag{12}$$

Equation (12) is a polynomial of degree four in λ and therefore has, in general, four complex roots. Let these roots be designated by $\lambda_j; j = 1, 2, \ldots, 4$. Then,

$$\psi(\xi) = \sum_{j=1}^{4} A_j e^{i\lambda_j\xi},$$

which may now be substituted into the boundary conditions to yield four homogeneous equations for four constants A_j. These equations are

$$\begin{aligned}\sum_{j=1}^{4} A_j &= 0,\\ \sum_{j=1}^{4} \lambda_j A_j &= 0,\\ \sum_{j=1}^{4} \lambda_j^2 A_j e^{i\lambda_j} &= 0,\\ \sum_{j=1}^{4} (\lambda_j^2-\eta)\lambda_j A_j e^{i\lambda_j} &= 0,\end{aligned} \tag{13}$$

where $\eta = F(2-\alpha^2/2)-\kappa$. System (13) has nontrivial solutions if and only if the determinant of the coefficients of $A_j; j = 1, 2, \ldots, 4$ is identically zero, i.e. the frequency equation is

$$\begin{aligned}\Delta \equiv\ & e^{i(\lambda_1+\lambda_2)}(\lambda_1^2\lambda_2^2+\eta\lambda_1\lambda_2)(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)\\ &-e^{i(\lambda_1+\lambda_3)}(\lambda_1^2\lambda_3^2+\eta\lambda_1\lambda_3)(\lambda_3-\lambda_1)(\lambda_4-\lambda_2)\\ &+e^{i(\lambda_1+\lambda_4)}(\lambda_1^2\lambda_4^2+\eta\lambda_1\lambda_4)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)\\ &+e^{i(\lambda_2+\lambda_3)}(\lambda_2^2\lambda_3^2+\eta\lambda_2\lambda_3)(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)\\ &-e^{i(\lambda_2+\lambda_4)}(\lambda_2^2\lambda_4^2+\eta\lambda_2\lambda_4)(\lambda_4-\lambda_2)(\lambda_3-\lambda_1)\\ &+e^{i(\lambda_3+\lambda_4)}(\lambda_3^2\lambda_4^2+\eta\lambda_3\lambda_4)(\lambda_4-\lambda_3)(\lambda_2-\lambda_1) = 0,\end{aligned} \tag{14}$$

where λ_1, λ_2, λ_3, and λ_4 are defined as functions of ω through equation (12).

3.2 *Torsional buckling*

To obtain the condition for divergent torsional motion, we let $\omega = 0$ in equation (12) and obtain $\lambda_{1,2} = 0$, and $\lambda_{3,4} = \pm\sqrt{(2F-\kappa)}$. Then, with $\kappa = \delta\pi^2$ and $\bar{F} = 2F-\kappa = \gamma\pi^2$, equation (14) reduces to

$$\alpha^2 = -\frac{4\gamma\cos\pi\sqrt{\gamma}}{(\gamma+\delta)(1-\cos\pi\sqrt{\gamma})}, \tag{15}$$

which is identical to the equation obtained by the authors for the torsional buckling of a cantilevered beam subjected at the free end to follower forces [1]. The first branch of the torsional buckling, corresponding to the first mode of instability, is shown by the solid line in Fig. 2.

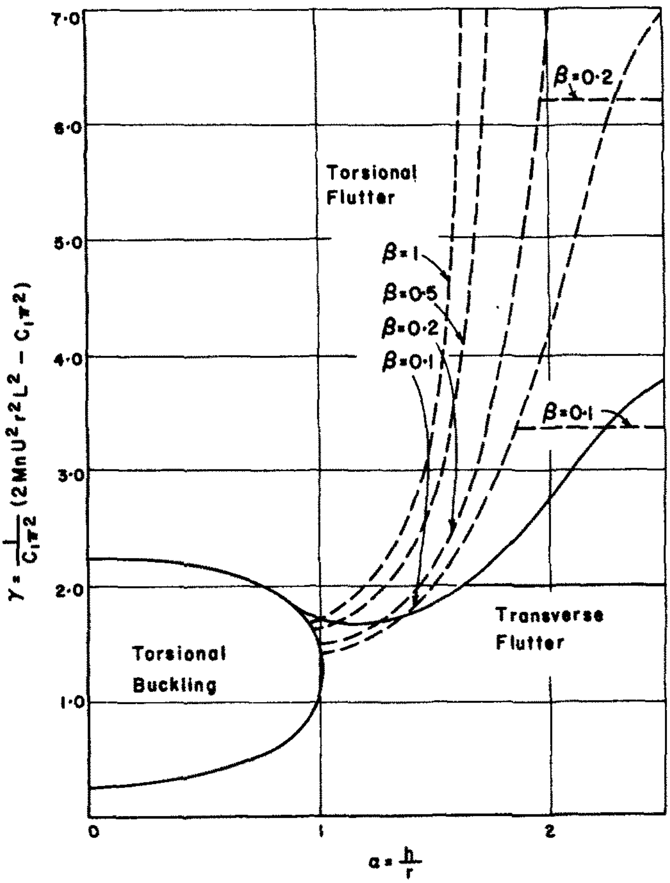

FIG. 2

3.3 *Torsional flutter*

For given α, β, n, δ and $\bar{F} = \gamma\pi^2$, equations (12) and (14) yield the frequencies of torsional oscillations. When $\bar{F}$ is small, these frequencies are all located on the left-hand side of the imaginary axis in the complex $i\omega$ plane and the system can perform only damped torsional oscillations.

As we increase $\bar{F}$, one of these frequencies approaches the imaginary axis, and for a certain value of $\bar{F}$, say $\bar{F}_{cr}$, equations (12) and (14) yield a real value for ω. If we now increase $\bar{F}$ beyond this critical value, one of the roots of (14) becomes complex with negative imaginary part. The beam will oscillate with an exponentially increasing amplitude. Consequently, we shall seek, for given α, β, n, and δ, values of ω (real) and $\bar{F}$ which identically satisfy (12) and (14). This can be done directly with the aid of a computer. The computer can be instructed to find the roots of equation (12) for specified values of α, β, n, δ, ω and γ, and then compute the value of Δ. By varying the value of ω and γ systematically, the critical ω and γ may easily be selected which make both real and imaginary parts of Δ identically zero. This is illustrated in Fig. 3 where for $\alpha = 1{\cdot}50$, $\delta = 1{\cdot}0$, $\beta = 1{\cdot}0$, and $n = 1$, both real and imaginary parts of $\Delta = \Delta_1 + i\Delta_2$ are plotted against the values of ω^2. We see that for $\gamma = 3{\cdot}40$, and $\omega^2 = 1{\cdot}13\pi^4$, Δ is identically zero. Similar results may

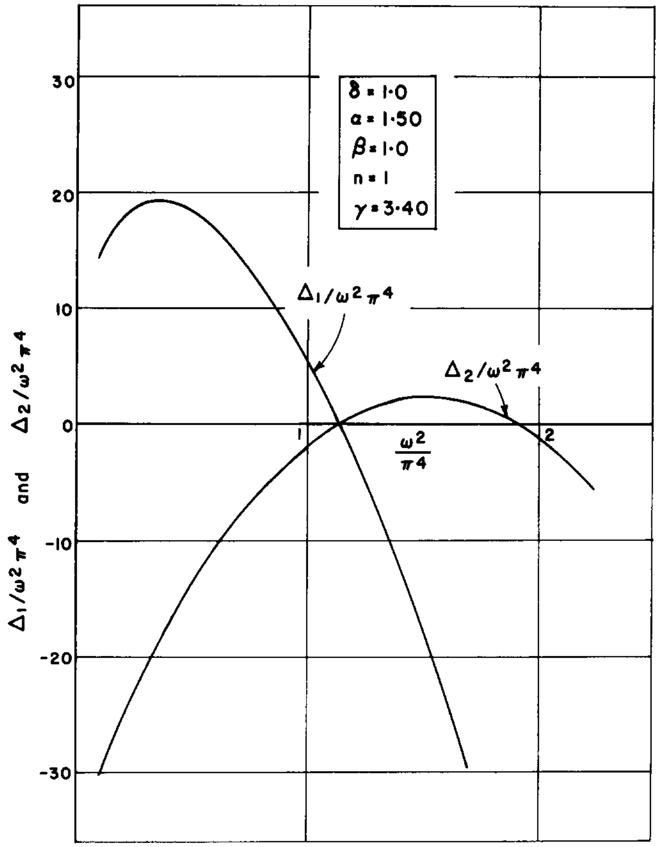

FIG. 3

be obtained for other values of α, β, and n. In this manner torsional flutter curves may be constructed. The first branch (the only practically significant one) of torsional flutter is shown in Fig. 2 by dashed lines, for $\delta = 1$, $n = 1$, and indicated values of β. The solid curve for torsional flutter in Fig. 2 is the limiting case when $n = \infty$ and corresponds to the torsional flutter of a cantilevered bar subjected at the free end to compressive follower forces [1].

It must be noted that, even for relatively large values of β ($n = 1$), the Coriolis forces may have a destabilizing effect for certain values of α. (For example, for $\beta = 0{\cdot}1$ and $1{\cdot}0 < \alpha < 1{\cdot}35$, as is seen in Fig. 2.)

3.4 *Transverse flutter*

In addition to torsional buckling and torsional flutter, the bar may lose stability also by transverse flutter [12]. The equation of motion and the boundary conditions for this case have been derived by employing Hamilton's principle in [9] and D'Alembert's principle in [12]. Here, we may simply identify C_1 with EI, $\varphi(z, t)$ with $y(z, t)$ and write

$$EI\frac{\partial^4 y}{\partial z^4}+2MnU^2\frac{\partial^2 y}{\partial z^2}+4MU\frac{\partial^2 y}{\partial z\,\partial t}+(2M+m)\frac{\partial^2 y}{\partial t^2} = 0,$$

$$y = \frac{\partial y}{\partial z} = 0; \quad \text{at} \quad z = 0$$

$$\frac{\partial^2 y}{\partial z^2} = \frac{\partial^3 y}{\partial z^3} = 0; \quad \text{at} \quad z = 0$$

which, by introducing the following dimensionless quantities:

$$\xi = \frac{z}{L}, \quad \tau = t\sqrt{\left[\frac{EI}{(2M+m)L^4}\right]}$$

$$F_1 = \frac{MnU^2L^2}{EI}, \qquad \beta = \frac{M}{m}, \qquad \beta'' = \frac{2\beta}{2\beta+1}$$

reduces to

$$\frac{\partial^4 y}{\partial \xi^4} + 2F_1\frac{\partial^2 y}{\partial \xi^2} + \sqrt{\left(\frac{8\beta''}{n}F_1\right)}\frac{\partial^2 y}{\partial \xi \partial \tau} + \frac{\partial^2 y}{\partial \tau^2} = 0,$$

$$y = \frac{\partial y}{\partial \xi} = 0; \quad \xi = 0$$

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^3 y}{\partial \xi^3} = 0; \quad \xi = 1.$$

Equation (12) now becomes

$$\lambda^4 - 2F_1\lambda^2 - \omega\sqrt{\left(\frac{8\beta''}{n}F_1\right)}\lambda - \omega^2 = 0, \tag{12'}$$

and equation (14) takes on the form

$$\begin{aligned}\Delta = {} & e^{i(\lambda_1+\lambda_2)}\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1)(\lambda_4-\lambda_3) \\ & - e^{i(\lambda_1+\lambda_3)}\lambda_1^2\lambda_3^2(\lambda_3-\lambda_1)(\lambda_4-\lambda_3) \\ & + e^{i(\lambda_1+\lambda_4)}\lambda_1^2\lambda_4^2(\lambda_4-\lambda_1)(\lambda_4-\lambda_2) \\ & + e^{i(\lambda_2+\lambda_3)}\lambda_2^2\lambda_3^2(\lambda_3-\lambda_2)(\lambda_4-\lambda_1) \\ & - e^{i(\lambda_2+\lambda_4)}\lambda_2^2\lambda_4^2(\lambda_4-\lambda_2)(\lambda_3-\lambda_1) \\ & + e^{i(\lambda_3+\lambda_4)}\lambda_3^2\lambda_4^2(\lambda_4-\lambda_3)(\lambda_2-\lambda_1) = 0.\end{aligned} \tag{14'}$$

For a given β and n, we now seek values of ω and F_1 which identically satisfy (12′) and (14′). In this manner we obtain the limit of transverse flutter, as shown by horizontal dashed lines in Fig. 2 for $EIr^2/C_1 = 1{\cdot}5$ and $\beta = 0{\cdot}1$, $0{\cdot}2$. In this figure, the horizontal solid line indicates the limit of transverse flutter for $n = \infty$ [1]. We note that for $\beta = 0{\cdot}5$, $1{\cdot}0$, the transverse flutter occurs at $\gamma = 12{\cdot}2$, and $15{\cdot}8$ respectively. These values are not shown in Fig. 2.

4. ANALYSIS OF FLUTTER BY INDIRECT METHOD

The method used in the previous section for the analysis of flutter was a direct one. That is, for a given system we directly obtained the critical values of γ and ω. One may solve the same problem by an indirect method which was employed in [12].

To this end we let λ_j; $j = 1, 2, \ldots, 4$ denote the roots of (12). Then we have

$$
\begin{aligned}
\lambda_1+\lambda_2+\lambda_3+\lambda_4 &= 0,\\
\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4 &= -(2F-\kappa),\\
\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4 &= \sqrt{\left(\beta'\frac{F}{n}\right)}\alpha^2\omega,\\
\lambda_1\lambda_2\lambda_3\lambda_4 &= -\omega^2.
\end{aligned}
\tag{16a}
$$

The first equation in (16a) is identically satisfied if we let

$$
\begin{aligned}
\lambda_1 &= a-b-c,\\
\lambda_2 &= -a+b-c,\\
\lambda_3 &= -a-b+c,\\
\lambda_4 &= a+b+c,
\end{aligned}
\tag{16b}
$$

and from the remaining equations we obtain

$$
\begin{aligned}
a^2+b^2+c^2 &= \frac{2F-\kappa}{2},\\
abc &= \frac{1}{8}\sqrt{\left(\beta'\frac{F}{n}\right)}\alpha^2\omega,\\
a^4+b^4-2a^2b^2-2b^2c^2-2c^2a^2 &= -\omega^2.
\end{aligned}
\tag{16c}
$$

We now let

$$
\begin{aligned}
a &= \tfrac{1}{2}(p+iq),\\
b &= \tfrac{1}{2}(p-iq),
\end{aligned}
\tag{16d}
$$

and from (16c) obtain

$$
\begin{aligned}
p^2-q^2+2c^2 &= 2F-\kappa,\\
(p^2+q^2)c &= \frac{1}{2}\sqrt{\left(\beta'\frac{F}{n}\right)}\alpha^2\omega,\\
(p^2-c^2)(q^2+c^2) &= \omega^2,
\end{aligned}
\tag{16e}
$$

where p, q, and c are all real. Substituting from (16d) into (16b) and then into the frequency equation (14), we finally arrive at, after a series of tedious manipulations,

$$\Delta \equiv \Delta_1 + i\Delta_2 = 0,$$

where:

$$\begin{aligned}\Delta_1 = {} & cp\{2[3p^2q^2-q^4-c^2(p^2+q^2)+4c^4]+(p^2-3q^2-4c^2)\eta\}\cos p \sinh q \\ & +cq\{2[p^4-3p^2q^2-c^2(p^2+q^2)-4c^4]-(3p^2-q^2-4c^2)\eta\}\sin p\cosh q \\ & +pq\{[(p^2+q^2)(q^2-p^2+2c^2)]+(p^2+q^2)\eta\}\sin 2c, \\ \Delta_2 = {} & \{[p^2q^2(q^2-p^2)+c^2(p^4+q^4-6p^2q^2)-3c^4(p^2-q^2)-4c^6] \\ & +[2p^2q^2-c^2(p^2-q^2)+4c^4]\eta\}\sin p\sinh q \\ & +pq\{2[-p^2q^2+3c^2(p^2-q^2)-7c^4]-[p^2-q^2-2c^2]\eta\}\cos p\cosh q \\ & -pq\{[p^4+q^4+2c^2(q^2-p^2)+2c^4]-[p^2-q^2-2c^2]\eta\}\cos 2c.\end{aligned} \tag{17}$$

For an assumed value of c and given α and $\kappa = \delta\pi^2$, we may now find p and q such that $\Delta_1 = \Delta_2 = 0$. Then, from equations (16e) the corresponding values of F, β, and ω, for a given n, may be computed.

The above method is an indirect one, as we do not know, in advance, which particular problem is being investigated. Moreover, if a computer is being used to find values of p and q which satisfy $\Delta_1 = \Delta_2 = 0$, it is then just as easy to employ the direct method outlined in the previous section. However, for small values of Coriolis forces, that is for sufficiently small $\sqrt{(\beta'/n)}$, one may reduce equations (17) by neglecting the higher order terms in c and study the effect of small Coriolis forces directly. This we shall discuss in the following section.

5. THE EFFECT OF SMALL CORIOLIS FORCES

We consider equation (17) and by neglecting $O(c^2)$ and higher order terms obtain

$$\begin{aligned}\bar{\Delta}_1 = {} & p\{2[3p^2q^2-q^4]+[p^2-3q^2]\eta\}\cos p\sinh q \\ & +q\{2[p^4-3p^2q^2]+[q^2-3p^2]\eta\}\sin p\cosh q \\ & +2pq\{(q^4-p^4)+(p^2+q^2)\eta\} = 0, \\ \bar{\Delta}_2 = {} & pq[(q^2-p^2)+2\eta]\sin p\sinh q \\ & +[-2p^2q^2-(p^2-q^2)\eta]\cos p\cosh q \\ & -[p^4+q^4-(p^2-q^2)\eta] = 0,\end{aligned} \tag{18}$$

where

$$\begin{aligned} p^2 &= \sqrt{\left(\omega^2+\bar{F}\frac{2}{4}\right)}+\frac{\bar{F}}{2} \\ q^2 &= \sqrt{\left(\omega^2+\bar{F}\frac{2}{4}\right)}-\frac{\bar{F}}{2}.\end{aligned} \tag{19}$$

The second equation in (18) is the frequency equation for $n = \infty$, (no Coriolis forces [1]), and the first equation, to the first order of approximation in $\sqrt{(\beta'/n)} = O(c)$, presents the effect of sufficiently small Coriolis forces. We note that $\bar{\Delta}_1$ and $\bar{\Delta}_2$ are both independent of c and, therefore, we may directly seek values of ω and $\bar{F}$ which make them identically zero. This is illustrated in Fig. 4 for $\alpha = 1{\cdot}5$, where the critical load is found to be $\gamma = 1{\cdot}67$. In Fig. 5, the critical load γ is plotted against α for sufficiently small Coriolis forces (the dashed curve). The solid curve for torsional flutter in this figure is for the limiting case of $n = \infty$ [1]. We note that the existence of Coriolis forces does not alter the region of divergent motion, as is expected. However, it makes this region a closed set—that is, in the presence of Coriolis forces, the points on the divergent curve indicate neutrally stable states. The horizontal solid line in Fig. 5 denotes the limit of transverse flutter for $n = \infty$, and the horizontal dashed line indicates that limit for sufficiently small Coriolis forces [7], (for $EI_x r^2/C_1 = 1{\cdot}5$).

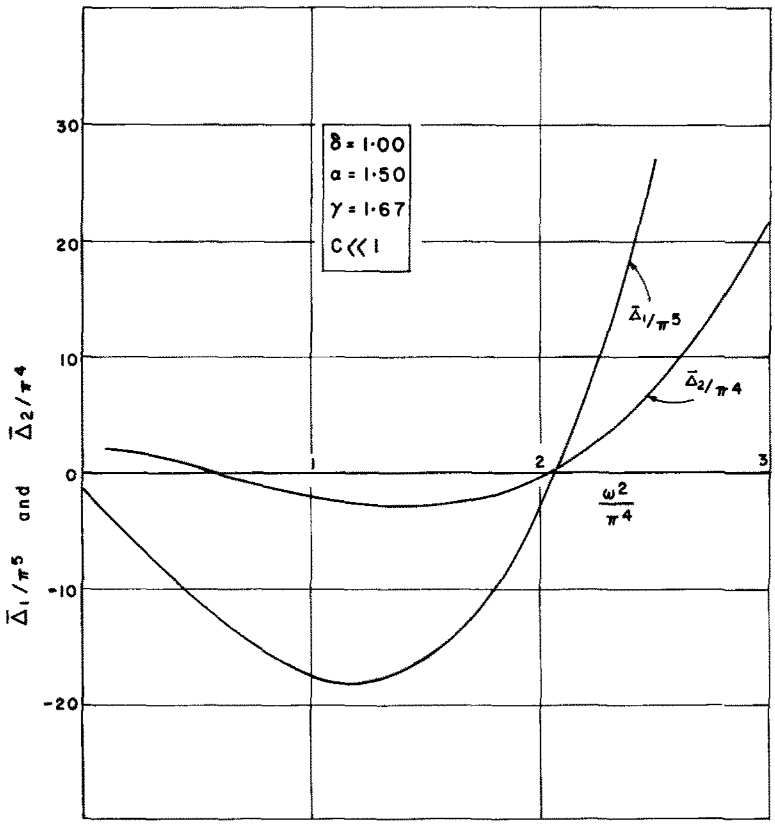

FIG. 4

It may be of interest to obtain the critical values of γ for $\beta = \infty$ and $n = 1$. This, of course, provides the upper limit of torsional and transverse flutter. The dotted curve in Fig. 5 represents this limiting case for $\delta = 1$. We note that transverse flutter, for $\beta = \infty$ and $n = 1$, occurs at $\gamma \approx 47$, which is not shown in Fig. 5.

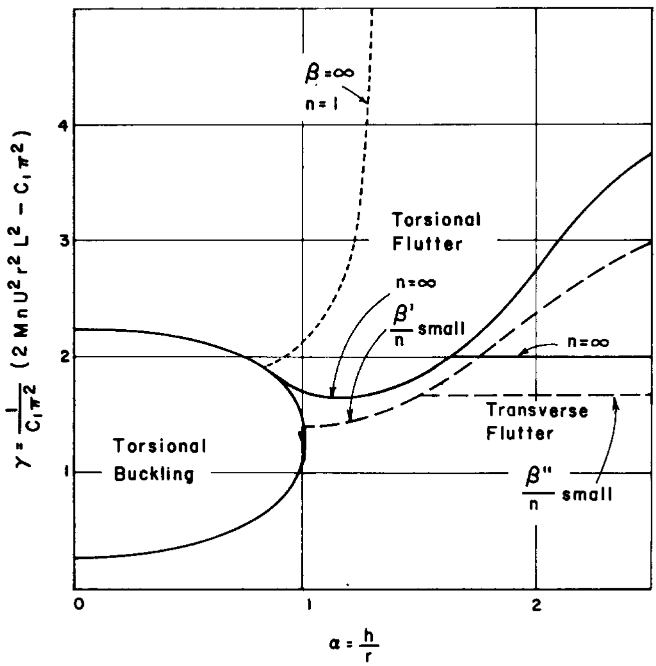

FIG. 5

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Résumé—Dans cette étude, une barre à console à section constante et ayant des tuyaux à fluide flexibles est examinée. On remarque que pour certaines sections transversales de la barre l'on peut perdre toute stabilité soit par divergence de torsion (flambement de torsion) soit par perturbation vibratoire de torsion selon la position des tuyaux en relation au centre de gravité de la section transversale. De plus, une perturbation vibratoire pourrait advenir mais pas un flambement transversal. L'on remarque de plus que les forces de Coriolis, présentes à cause du mouvement du fluide dans les tuyaux vibrants, pourraient avoir soit un effet stabilisant soit un effet instabilisant, selon les paramètres du système.

Zusammenfassung—Ein Kragträger Stab mit gleichmäßigem Querschnitt der biegsame Röhren trägt die Flüssigkeiten fördern wird untersucht. Es wird gezeigt, daß für bestimmte Stabquerschnitte die Stabilität entweder durch Torsionsknicken oder durch Torsionsflattern verloren werden kann, je nachdem wo die Rohre angebracht sind mit Hinsicht auf den Schwerpunkt des Querschnittes. Ferner kann auch ein Transversalflattern entstehen aber kein Transversalknicken. Weiters wird gezeigt, daß die Coriolis-Kräfte, durch die Flüssigkeitsbewegung in vibrierenden Rohren bedingt, je nach Umständen die Stabilität entweder positiv oder negativ beeinflußen.

Абстракт—Обсуждается консольная балка однородного поперечного сечения с прикреплёнными к ней гибкими трубами, подающими жидкость. Показано, что для некоторых поперечных сечений балки устойчивость может быть потеряна крутильным расхождением (крутильным изгибанием) или крутильной вибрацией (флайтер), зависящими от местоположения труб по отношению к пентру тяжести поперечного сечения. В дополнение может получиться также поперечная вибрация, но не может получиться поперечного изгибания. Далее показано, что Кориольные силы (Coriolis) которые присутствуют благодаря движению жидкости в вибрирующих трубах могут иметь стабилизирующий или дестабилизирующий эффект, зависящий от параметров системы.